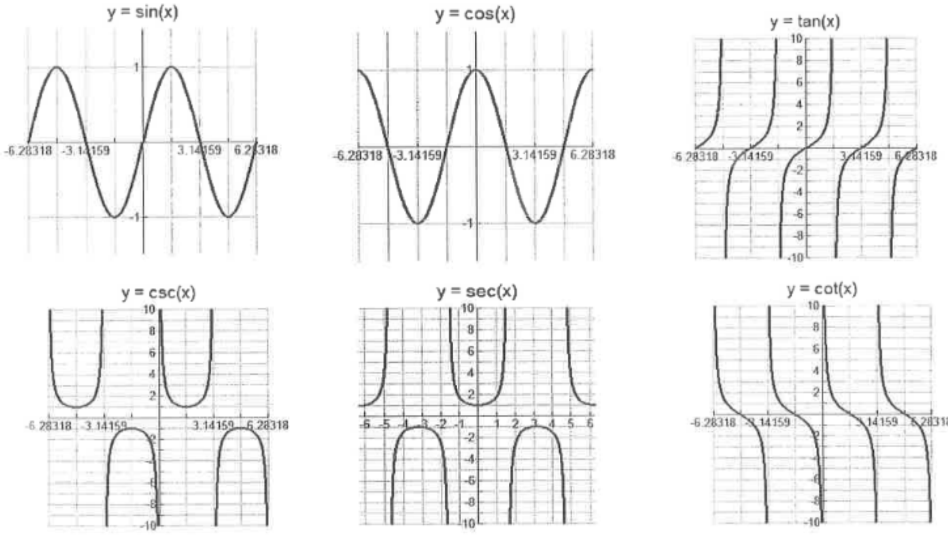
AP Calculus AB
Lesson 4-3: *Derivatives of Inverse Functions, Part 2*

Name Heml 2016
Date

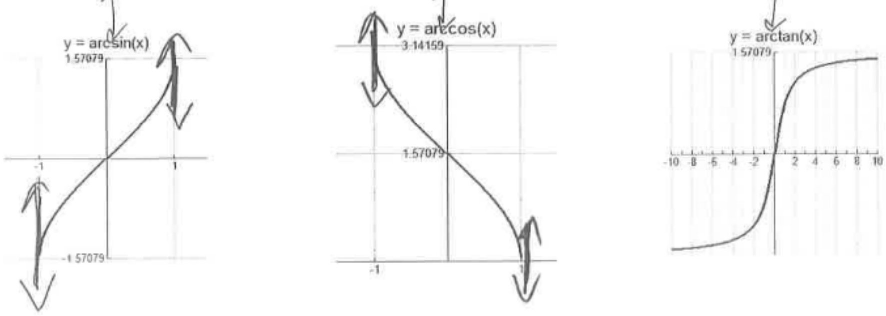
Learning Goal:

- *I can calculate the derivative of an inverse trigonometric function.*

Let's take a trip down our trigonometric memory lane . . . . .



$y = \sin^{-1}(x)$ $\quad$ $y = \cos^{-1}(x)$ $\quad$ $y = \tan^{-1}(x)$



Look at the graph of $y = \sin x$ on the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The sine function is one-to-one on this interval and hence it has inverse $y = \sin^{-1} x$. We know from the previous lesson that we are assured that the inverse function $y = \sin^{-1} x$ is differentiable throughout the interval $-1 < x < 1$ (it is not differentiable at $x = 1$ or $x = -1$ because the tangent lines are vertical at these points).

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We find the derivative of $y = \sin^{-1} x$ as follows:

$$y = \sin^{-1} x$$

$x = \sin y$ (def. of inverse)

$\frac{d}{dx}(x) = \frac{d}{dx}(\sin y)$

$1 = (\cos y)\frac{dy}{dx}$

$\frac{1}{\cos y} = \frac{dy}{dx}$ ← $\cos y \neq 0$ for $\frac{-\pi}{2} < y < \frac{\pi}{2}$ so $\cos y > 0$ [positive]

Recall: $\sin^2 y + \cos^2 y = 1$

$\cos^2 y = 1 - \sin^2 y$

$\cos y = \pm\sqrt{1 - \sin^2 y}$

$\frac{1}{\sqrt{1-\sin^2 y}} = \frac{dy}{dx}$

$(x)^2 = (\sin y)^2$

$\frac{1}{\sqrt{1 - x^2}} = \frac{dy}{dx}$

$\therefore \frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ $\Rightarrow$ $\boxed{\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}}$

$dx(\quad) \quad \sqrt{1-u^2}$

$= \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x$

$\boxed{= \frac{2x}{\sqrt{1-x^4}}}$

Without going through the pain of proving all of the other formulas, here they are:

$$\frac{d}{dx}\sin^{-1}(u)=\frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx} \qquad \frac{d}{dx}\cos^{-1}(u)=-\frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\tan^{-1}(u)=\frac{1}{1+u^2}\cdot\frac{du}{dx} \qquad \frac{d}{dx}\cot^{-1}(u)=-\frac{1}{1+u^2}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\sec^{-1}(u)=\frac{1}{|u|\sqrt{u^2-1}}\cdot\frac{du}{dx} \qquad \frac{d}{dx}\csc^{-1}(u)=-\frac{1}{|u|\sqrt{u^2-1}}\cdot\frac{du}{dx}$$

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Let's have some fun . . . and practice our inverse trigonometric derivatives!!
Differentiate the following functions:

1. $f(x)=\arcsin(2x+3)$

$u=2x+3 \quad \frac{du}{dx}=2$

$$f'(x)=\frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$$

$$=\frac{1}{\sqrt{1-(2x+3)^2}}\cdot 2=\boxed{\frac{2}{\sqrt{1-(2x+3)^2}}}$$

2. $g(x)=e^{\arctan x}$

$u=\arctan x$

$$g'(x)=e^u\cdot\frac{du}{dx}$$

$$=e^{\arctan x}\cdot\frac{1}{1+x^2}=\boxed{\frac{e^{\arctan x}}{1+x^2}}$$

3. $h(x)=x\sec^{-1}(3x)$ $\quad \sec^{-1}(3x) \quad u=3x \quad \frac{du}{dx}=3$

$$h'(x)=x\cdot\frac{1}{3x\sqrt{9x^2-1}}\cdot 3+\operatorname{arcsec}(3x)\cdot 1$$

$$=\frac{\cancel{3x}}{\cancel{3x}\sqrt{9x^2-1}}+\operatorname{arcsec}(3x)$$

$$=\frac{1}{\sqrt{9x^2-1}}+\operatorname{arcsec}(3x)$$

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